Degree of Separability of Bipartite Quantum States

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Review
Lewenstein–Sanpera decompositions

- Given $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$
- Q: How quantum is $\rho$?
- LSD [Lewenstein, 98]: convex decomposition of $\rho$ into a separable piece and a positive remainder

$$\rho = \lambda \rho_{\text{sep}} + (1 - \lambda) \sigma_{\text{ent}}$$

- Compactness of set of separable states $\Rightarrow$ optimal decomposition with $S = \max \{\lambda\}$ exists.
- Optimal LSD denoted by

$$\rho = S \rho_{\text{sep}} + (1 - S) \sigma_{\text{ent}} \equiv \tilde{\rho}_{\text{sep}} + \tilde{\sigma}_{\text{ent}}$$
Lewenstein–Sanpera decompositions

- Optimal LSD $\rho = S_{\rho_{\text{sep}}} + (1 - S)\varsigma_{\text{ent}}$ is unique
- Identify $S$ as the degree of separability of $\rho$
- Identify $\rho_{\text{sep}}$ as the best-separable approximation (BSA) to $\rho$
- Constrained convex optimization problem in variable $\tilde{\rho}_{\text{sep}} \in \mathbb{H}^{n \times n} \simeq \mathbb{R}^{n^2}$, $n = d_A d_B$
- Maximize linear objective function $\text{tr}\{\tilde{\rho}_{\text{sep}}\}$
- Convex constraints:
  - $\rho_{\text{sep}}$ is separable $\rightarrow$ difficult
  - $\sigma_{\text{ent}} = \rho - \tilde{\rho}_{\text{sep}} \geq 0 \rightarrow$ easier
Semidefinite programming

- The primal semidefinite program in $m$ variables is:

  \[
  \text{minimize} \quad \vec{c}^T \vec{x} \\
  \text{subject to} \quad F(\vec{x}) \geq 0,
  \]

  where $F(\vec{x}) = F_0 + \sum_{i=1}^m x_i F_i, \vec{x} \in \mathbb{R}^m, F_0, F_i \in \mathbb{H}^{n \times n}$

- The dual problem is

  \[
  \text{maximize} \quad -\text{tr}\{F_0 Z\} \\
  \text{subject to} \quad \text{tr}\{F_i Z\} = c_i, \quad i = 1, \ldots, m, \\
  Z \geq 0.
  \]

  where $0 \leq Z = Z^\dagger \in \mathbb{H}^{n \times n}_+$
Semidefinite programming

- Efficient and reliable SDP solvers
  - SDPT3, SeDuMi, SDPA etc.
- Powerful duality theory
  - Can usually get strict bounds from primal and dual problems
  - Optimality conditions
  - Geometrical interpretation
- Has appeared naturally in various contexts in QIT
  - distillable entanglement, CP maps, USD, EWs, entanglement measures
- We will show how to employ SDP to tackle the optimal LSD problem
Optimal LSD with symmetric extensions
A k-Bose symmetric extension (k-BSE) of $\rho$ is a linear operator $\tilde{\rho}_k \in B(\mathcal{H}_A \otimes \mathcal{H}_B^\otimes k)$ fulfilling the following:

- $\tilde{\rho}_k$ is positive semidefinite
- $\text{tr}_{B^{k-1}}\{\tilde{\rho}_k\} = \rho$
- $\tilde{\rho}_k(\mathbb{I}_A \otimes \Pi_{\text{symm}}^k) = \tilde{\rho}_k$, where $\Pi_{\text{symm}}^k$ denotes the projector onto the symmetric subspace of $\mathcal{H}_B^\otimes k$

A k-PPT-BSE of $\rho$ additionally satisfies

$$\tilde{\rho}_k^{T_{B^{[k/2]}}} \geq 0$$

$\rho$ is separable iff it admits a k-(PPT)-BSE for all $k \in \mathbb{N}$: DPS-(PPT) [Doherty, 04] criterion for separability
Let $S_{(p)}^k$ denote the convex cone of linear operators admitting a k-(PPT)-BSE. Then $\{S_{(p)}^k\}_{k=1}^\infty$ approximates $S$ from the outside:

$$S_{(p)}^1 \supseteq S_{(p)}^2 \supseteq \ldots \supseteq S$$

$$\lim_{k \to \infty} S_{(p)}^k = \bigcap_{k=1}^\infty S_{(p)}^k = S$$

Main point: separability constraint can be phrased as a (countably infinite) sequence of positivity constraints.
The primal semidefinite program has the following form:
\[
\begin{align*}
\text{minimize} & \quad \vec{c}^T \vec{x} \\
\text{subject to} & \quad F(\vec{x}) \geq 0,
\end{align*}
\]

**Symmetric extensions**

- Optimal LSD with symmetric extensions
- Low-rank states
- Conclusion

**Review**

- Approximation from outside
- Approximation from inside
- Numerics

**Symmetric extensions - Geometry**

\( S \quad \ldots \quad S^2_{(p)} \quad S^1_{(p)} \)
Optimal LSD via symmetric extensions

For each $k \in \mathbb{N}$, one can ask for the $k$-(PPT)-optimal decomposition

$$\rho = \lambda_k \varrho_k + (1 - \lambda_k) \varsigma_k \equiv \tilde{\varrho}_k + \tilde{\varsigma}_k$$

Optimization over the variable $\bar{\rho}_k \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B^{\otimes k})$

Maximization of a linear objective function:

$$\text{tr}\{\tilde{\rho}_k\} = \text{tr}_{AB}\{\text{tr}_{B^{k-1}}\{\bar{\rho}_k\}\} = \text{tr}\{\bar{\rho}_k\}$$

Subject to positivity constraints:

$$\bar{\rho} \geq 0$$

$$\tilde{\sigma}_k = \rho - \tilde{\rho}_k = \rho - \text{tr}_{B^{k-1}}\{\bar{\rho}_k\} \geq 0$$

$$\bar{\rho}_{\frac{k}{2}}^T \geq 0$$
Therefore, finding the $k$-(PPT)-optimal decomposition is a SDP.

Obtain sequence of $k$-(PPT)-optimal decompositions

\[
\rho = \lambda_1 \varrho_1 + (1 - \lambda_1) \varsigma_1 \\
= \lambda_2 \varrho_2 + (1 - \lambda_2) \varsigma_2 \\
\vdots \\
= \lambda_k \varrho_k + (1 - \lambda_k) \varsigma_k \\
\vdots
\]

Q: Does this sequence converge to the true optimal LSD?
Optimal LSD via symmetric extensions

- **A:** Yes
  \[ \lambda^k \to S \text{ (from above)} \]
  \[ \varrho_k \to \varrho_{\text{sep}} \]
- Proof: Omitted.
- **Conclusion:** Although optimal LSD problem is not a SDP *per se*, we can access it by taking the limit of the $k$-(PPT)-optimal decompositions, which *can* be computed using SDP.
Optimal LSD via symmetric extensions - Geometry

- Unit trace section
- Negative semidefinite cone
- Symmetric extensions
- Approximation from outside
- Approximation from inside
- Numerics

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Degree of Separability of Bipartite Quantum States
Optimal LSD via symmetric extensions

- DPS-criterion approximates $S$ from the **outside**: gives upper bounds for $S$.
- DPS*-criterion [Navascués, 09] approximates $S$ from the **inside**: gives lower bounds for $S$.
- Perturb the sets $S^k_p$ to get $S^*_k(p) \subseteq S$:

\[
S^*_k(p) = \left\{ \frac{k}{k + d_B} \sigma + \frac{d_B}{k + d_B} \text{tr}_B\{\sigma\} \otimes \frac{I_B}{d_B} : \sigma \in S^k_p \right\},
\]

\[
S^*_k(p) = \left\{ (1 - \epsilon_k)\sigma + \epsilon_k \text{tr}_B\{\sigma\} \otimes \frac{I_B}{d_B} : \sigma \in S^k_p \right\}
\]

- $S^*_k(p) \subseteq S$ for all $k$, and $\lim_{k \to \infty} S^*_k(p) = S$
  - Properties are less nice — no hierarchy
Optimal LSD via symmetric extensions

For each \( k \in \mathbb{N} \), obtain a \( k^*\)-(PPT)-optimal decomposition via SDP

\[
\rho = \lambda_1^* \rho_1^* + (1 - \lambda_1^*) \varsigma_1^* \\
= \lambda_2^* \rho_2^* + (1 - \lambda_2^*) \varsigma_2^* \\
\vdots \\
= \lambda_k^* \rho_k^* + (1 - \lambda_k^*) \varsigma_k^* \\
\vdots
\]

\textbf{Q:} Does this sequence converge to the true optimal LSD?

\textbf{A:} Yes, provided \( \rho \) has full-rank

\textbf{Proof:} omitted
For each $k \in \mathbb{N}$, obtain a $k^*$-(PPT)-optimal decomposition via SDP

$$
\rho = \lambda_1^* \varrho_1^* + (1 - \lambda_1^*) \varsigma_1^*
$$

$$
= \lambda_2^* \varrho_2^* + (1 - \lambda_2^*) \varsigma_2^*
$$

$$
\vdots
$$

$$
= \lambda_k^* \varrho_k^* + (1 - \lambda_k^*) \varsigma_k^*
$$

$$
\vdots
$$

**Q:** Does this sequence converge to the true optimal LSD?

**A:** Yes, provided $\rho$ has full-rank

**Proof:** omitted
Complexity considerations

- Size of SDP appears to increase exponentially with $k$
  - $\dim(\mathcal{H}_A \otimes \mathcal{H}_B^\otimes k) = d_A d_B^k$
  - No. of SDP variables $= (d_A d_B^k)^2$
  - No. of operations needed $\sim$ exponential in problem size

- Actually it suffices to consider linear operators on $\mathcal{H}_A \otimes \mathcal{H}_B^{\otimes k}_{\text{symm}}$, which has dimension $d_A \cdot \binom{d_B + k - 1}{k}$, polynomial in $k$

- Complexity of $k$-th SDP scales polynomially with $k$
Generic state in $\mathbb{C}^2 \otimes \mathbb{C}^2$
Generic state in $\mathbb{C}^4 \otimes \mathbb{C}^2$
BES in $\mathbb{C}^4 \otimes \mathbb{C}^2$. Reduced-rank (L), Full-rank (R)
Generic state in $\mathbb{C}^5 \otimes \mathbb{C}^2$
Generic state in $\mathbb{C}^3 \otimes \mathbb{C}^3$

The diagram illustrates the degree of separability of bipartite quantum states for different cases.

- **DPS-PPT**
- **DPS**
- **DPS*-PPT**
- **DPS***

The figure shows the evolution of $\lambda_{k_\epsilon}$ with respect to $k_\epsilon$ for various approximations and separability criteria.
Low-rank states
Optimal LSD of two-qubit states

- Optimal decompositions only known for a few classes of states
  - Wellens–Kuś [Wellens, 01] equations characterize some classes, but difficult to solve
  - General numerical approach was to subtract projectors onto product vectors in the range of $\rho$

- Separability $\equiv$ positivity of partial transpose
  - $S = \{\text{PPT states}\} = S^1_\rho = S^2_\rho = \ldots$

- Optimal LSD problem is *exactly* a SDP; no approximation required [Thiang, 09]
Optimal LSD of two-qubit states

- Strict primal and dual feasibility ⇒ Opt. conditions given by complementary slackness:
  \[(\mathcal{Z}_1 + \mathcal{Z}_2^{T_B})\tilde{\varsigma}_{\text{ent}} = -\tilde{\varsigma}_{\text{ent}}\]
  \[\tilde{\varrho}_{\text{sep}}^{T_B}\mathcal{Z}_2 = 0\]
  \[\tilde{\varrho}_{\text{sep}}\mathcal{Z}_1 = 0\]

- Equivalent to original WK equations, derived here in a transparent way
- \((\mathcal{Z}_1 + \mathcal{Z}_2^{T_B})\) is the “optimal” EW — decomposable
Optimal LSD of two-qubit states

- Reduced-rank case can be similarly handled with an appropriate re-parameterization
- Duality theory of SDP gives optimality conditions, and an efficient way to solve for the optimal LSD
- PPT criterion remains sufficient for separability in $\mathbb{C}^2 \otimes \mathbb{C}^3$
- Straightforward generalization of SDP formulation to qubit-qutrit case
Optimal LSD of rank-2 states

- Optimal decompositions known for rank-2 two-qubit states [Englert, 02]
- Regardless of Hilbert space dimension, no numerical optimization needs to be carried out for rank-2 states
- $\tilde{\rho}_{\text{sep}}$ is a sum of projectors onto product states in $\mathcal{R}(\rho)$
- Range of entangled $\rho_{\text{rank-2}}$ contains at most 2 product vectors
  - Simplifies analysis — at most two projectors to consider
- Every 2D-subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product vector
  - Abundance of 2D-subspaces of higher dimensional tensor product spaces that contain no product vectors
Conclusion
Conclusion

• Outlined a systematic way to obtain the optimal LSD of an arbitrary bipartite state in any finite dimension
  • Optimal LSD as the limit of a sequence of decompositions which are available via SDP
  • Complementary sequence via DPS*- (PPT) criterion also converges to the optimal LSD if \( \rho \) has full-rank

• Implemented and tested the procedure numerically
  • Excellent bounds obtained for \( S \) are obtained for qudit-qubit states
  • Inclusion of PPT constraint increases rate of convergence (w.r.t. \( k \)), but is computationally more expensive
  • More extensions can be considered if PPT constraint is dropped — can give better lower bounds
Conclusion

- Cast the optimal LSD problem for two-qubits and qubit-qutrits as a SDP
  - Used duality to obtain optimality conditions
- Provided analytically the optimal LSD of rank-2 states in any dimension
- For details, see arXiv:1005.3675
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END
References


